H∞ ROBUST CONTROL OF FLEXIBLE MANIPULATOR VIBRATION BY USING A PIEZOELECTRIC-TYPE SERVO-DAMPER

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Abstract. An H∞ controller design scheme based on the matrix fraction stability condition is proposed for the vibration control of a flexible manipulator, which has the matrix fraction uncertainty. The proposed controller is then applied to the cantilever beam with a piezoelectric-type servo-damper so that the vibration at the free end of the beam is effectively suppressed, while its effective length slowly changes.

Key Words. H∞ control; multivariable control system; piezoelectric; robots; robust control; stability criteria; vibration control

1. INTRODUCTION

Recently, the needs of light structures are growing in the field of industrial robots in order to increase the speed of motion and thus the productivity. As a slender structure gets lighter, it tends to be flexible, leading to severe operational vibration at its free end. In the design of a vibration controller for a system such as a multi-axis robot arm or a manipulator with varying length, the robust stability should first be checked since the dynamics also changes during operation.

Unidirectional vibration control of a cantilever beam with a fixed length by using a piezoelectric-type servo-damper was investigated by Lee et al. (1991). As an extension of the previous work, the horizontal and vertical directional vibration control of a cantilever beam was performed while its effective length slowly changes (Lee et al., 1992). Though these works for the cantilever beam achieved good performances by integral control action, robust stability was not considered. In this study, H∞ control theory is adopted as the design technique for a robust controller, which robustly stabilizes the system and satisfies the desired performance, since it has two advantages: One is the fact that variations in system parameters can be considered as the uncertainty in a mathematical model, requiring, not an accurate model of uncertainty, but only its upper bound. The other is that the desired performance of the system is defined as the maximal magnitude of the closed-loop transfer function which can be easily understood. In particular, H∞ control proves highly efficient for problems such as vibration isolation and vibration control in a mechanical system, where the design specification is, in general, given in the frequency domain.

In this study, an H∞ controller design scheme based on the matrix fraction stability condition is proposed for the vibration control of a flexible manipulator, which has the matrix fraction uncertainty. The matrix fraction stability condition gives a good robust stability by separating the variations in the numerator and denominator of the nominal plant. The proposed controller is then applied to the cantilever beam with a piezoelectric-type servo-damper so that the vibration at the free end of the beam is effectively suppressed while its effective length slowly changes. Vibration control experiments are performed to investigate the performance and robust stability of the proposed controller when the cantilever beam rests and moves.

2. H∞ ROBUST CONTROL

2.1. Design of H∞ Controller

The H∞ control problem has drawn much attention in the area of control system design since Zames
(1981) initiated this problem. Several developments achieved in $H_\infty$ control theory are well summarized in (Francis, 1987; McFarlane and Glover, 1989; Doyle et al., 1992). Recently, Doyle et al. (1989) developed a theory for regulated output feedback control such that the controller is obtained by solving two algebraic Riccati equations, but the assumptions are quite restrictive. For general output feedback control, Sampei et al. (1990) derived the solvability condition by introducing two algebraic Riccati inequalities.

The basic system block diagram used in $H_\infty$ control theory is shown in Fig.1 where $G(s)$ is the generalized plant and $K(s)$ is the controller. The system is assumed to be finite-dimensional linear time-invariant. The signal $w(t)$ contains all external inputs, including disturbances, sensor noise and commands; the output $z(t)$ is the signals desired to be controlled, $y(t)$ is the measured variables, and $u(t)$ is the control inputs.

$$\begin{align*}
G(s) & \xrightarrow{w} z \\
& \xrightarrow{u} y \\
& \xrightarrow{K(s)} z
\end{align*}$$

Fig.1. Basic block diagram

In general, a generalized plant $G(s)$ is described, in the matrix form, as follows (Doyle et al., 1989):

\[
\begin{align*}
\xi(t) &= A\xi(t) + B_1w(t) + B_2u(t), \\
z(t) &= C_1\xi(t) + D_{11}w(t) + D_{12}u(t), \\
y(t) &= C_2\xi(t) + D_{21}w(t),
\end{align*}
\]

(1)

where $\xi(t)$ denotes the states. It is assumed that $(A, B_2)$ is stabilizable and $(C_2, A)$ is observable. Let $G_{\omega\omega}(s)$ be the closed-loop transfer function from $w(t)$ to $z(t)$ and $y$ be a prespecified positive scalar constant. The conditions for the existence of a strictly proper controller are then given by the following theorem.

**Theorem** (Sampei et al., 1990) There exists a strictly proper controller which stabilizes the system and satisfies $\|G_{\omega\omega}(s)\|_{\omega\omega} < \gamma$ if and only if the following conditions are satisfied.

i) $\gamma^2I - D_{11}^T D_{11} > 0$.

ii) There exist the symmetric matrices $P_c > 0$ and $P_o > 0$ which satisfy the following conditions: a) There exists $K_c$ such that

\[
P_cA_c + A_c^T P_c + (P_cB_2 + C_2^T D_{11})(\gamma^2I - D_{11}^T D_{11})^{-1}
\]

(2)

where $A_c = A + B_2K_c$, $C_c = C_1 + D_{12}K_c$.

b) There exists $K_o$ such that

\[
P_oA_0^T + A_oP_o + (P_oC_1^T + B_1D_{11}^T)(\gamma^2I - D_{11}^T D_{11})^{-1}
\]

\[
(P_oC_1^T + B_1D_{11}^T)^T + B_2B_2^T < 0
\]

(3)

where $A_0 = A + K_cC_2$, $B_0 = B_1 + K_oD_{21}$.

iii) $\gamma^2P_o^{-1} - P_o > 0$.

When these conditions hold, one of the strictly proper controllers with output feedback, for the case of $D_{11} = D_{12} = 0$, can be obtained using the following steps. These steps are less restrictive than those of Sampei and Mita (1990) since $Q_c$ and $Q_o$ are given as the absolute magnitudes of diagonal terms in equations (4) and (5), respectively, instead of the equivalent diagonal magnitudes.

**Step 1.** Initially, choose a sufficiently small diagonal matrix $J > 0$. $J$ is introduced to compensate the magnitude differences of diagonal terms. By letting

\[
K_c = -\frac{1}{2}J B_2^T P_c
\]

in equation (2), obtain $P_c > 0$ from the algebraic Riccati equation

\[
P_cA_c + A_c^T P_c + P_c(B_1B_1^T - B_1J B_2^T)P_c + C_2^T C_2 = 0
\]

(4)

for a sufficiently small diagonal matrix $Q_c > 0$. Diagonal elements of $Q_c$ are chosen as the absolute magnitudes of diagonal elements in the left term (error term) with $P_c$ obtained for $Q_c = 0$.

**Step 2.** By letting

\[
K_o = - (B_1D_1^T + P_oC_1^T)(D_21D_{11}^T)^{-1}
\]

in equation (3), obtain $P_o > 0$ from the algebraic Riccati equation

\[
P_oA_0^T + A_oP_o + P_o(C_1 + C_2^T (D_21D_{11}^T)^{-1}C_2)^T
\]

\[
+ D_{11}^T D_{11} + D_1^T D_1^{-1}D_1^T + B_2B_2^T = 0
\]

(5)

for a sufficiently small diagonal matrix $Q_o > 0$. Diagonal elements of $Q_o$ are chosen as the absolute magnitudes of diagonal elements in the left term (error term) with $P_o$ obtained for $Q_o = 0$.

**Step 3.** Verify if $\gamma^2P_o^{-1} - P_o > 0$ holds. If it does not hold, increase $J$ and retry Steps 1 and 3.

**Step 4.** When one of the strictly proper $H_\infty$ controllers with output feedback, $K(s)$, has the form of

\[
K(s) = C_k(sI - A_k)^{-1}B_k
\]

(6)
the state space realization matrices are given, without iteration from the results of the recursive steps 1 \sim 3, as (Sampei et al., 1990):

\[ C_k = K_c, \]
\[ B_k = -(I - \frac{1}{\gamma^2} P_e P_e^T)^{-1} K_o, \]
\[ A_k = A + B_k C_k - B_k C_2 - \frac{1}{\gamma^2} (B_k D_k - B_k) B_k^T P_e \]
\[ -(\gamma^2 P_e^T - P_e)^{-1} (C_k B_k P_e + Q_k). \]

2.2. Mixed Sensitivity Problem

Sensitivity and robust stability are the main issues in the \( H_\infty \) control theory. The sensitivity problem means the \( H_\infty \) optimization problem which minimizes the performance objectives, namely, the \( H_\infty \) norm. Robust stability is concerned with the stability of the plant with uncertainty. The mixed sensitivity problem, which combines the sensitivity and robust stability problems for a plant with uncertainty, is a typical \( H_\infty \) standard problem (Kwakernaak, 1985). In this section, the mixed sensitivity problem for the plant with additive uncertainty is discussed.

The system block diagram used in this study is shown in Fig.2, where \( G_o(s) \) and \( G_u(s) \) are the nominal transfer functions from \( w(t) \) and \( u(t) \) to \( y(t) \), respectively. The resulting closed-loop transfer function from \( w(t) \) to \( y(t) \), \( G_m(s) \), is defined as

\[ G_m(s) = [I - G_u(s) K(s)]^{-1} G_w(s), \]

from which \( L(s) \) is defined as

\[ L(s) = [I - G_u(s) K(s)]^{-1}. \]

Fig.2. Feedback system block diagram

The objective is to find the \( H_\infty \) controller \( K(s) \) which is robustly stable subject to uncertainty, and has the performance such that

\[ \| G_m(s) \|_\infty < \gamma. \]  

When uncertainties exist in the system, the closed-loop transfer function is given as

\[ G_m(s) = [I - G_u(s) K(s)]^{-1} G_w(s), \]

where \( G_m(s) \), \( G_u(s) \), and \( G_w(s) \) are the transfer functions that the uncertainties are added to \( G_m(s) \), \( G_u(s) \) and \( G_w(s) \), respectively. When \( G_u(s) \) has an additive uncertainty \( G_{u\Delta}(s) \), \( G_u(s) \) is expressed as

\[ G_u(s) = G_u(s) + G_{u\Delta}(s). \]

By substituting equation (12) into equation (11),

\[ G_m(s) = [I - G_u(s) K(s) - G_{u\Delta}(s) K(s)]^{-1} G_w(s). \]

Using equation (9), \( G_m(s) \) may be written as

\[ G_m(s) = L(s)[I - G_{u\Delta}(s) K(s) L(s)]^{-1} G_w(s). \]

A weighting function \( W_f(s) \) is chosen to satisfy

\[ \sigma[W_f(j\omega)] \geq \sigma[G_{u\Delta}(j\omega)] \quad \text{for all } \omega, \]

where \( \sigma[\bullet] \) and \( \sigma[\bullet] \) denote the minimum and maximum singular values, respectively. Assuming that \( G_u(s) \) is stable, the sufficient stability condition is given, by the small gain theorem (Desoer and Vidyasagar, 1975) from equation (14), as the \( H_\infty \) norm

\[ \left\| W_f(s) K(s) L(s) \right\|_\infty < 1. \]

The mixed sensitivity problem to satisfy both equations (10) and (16) is obtained as

\[ \left\| L(s) G_u(s) \right\|_\infty < \gamma. \]

2.3. Matrix Fraction Stability Condition

In general, the uncertainties are classified into the additive, multiplicative and coprime factor uncertainty models in the frequency domain, and the uncertainty model in the state-space realization (Khargonekar et al., 1990). Additive and multiplicative uncertainties (Doyle and Stein, 1981) are mainly used when the magnitude of the uncertainty is sufficiently small compared with the nominal plant. A coprime factor uncertainty model is more general than the corresponding classes allowed by additive and multiplicative uncertainty models. The idea of coprime factorization over proper, stable, real-rational functions (RHJ) is due to (Vidyasagar, 1972). The coprime factor
uncertainty model has been argued in (Vidyasagar, 1984). The robust stability condition for the plant with uncertainty was observed by Kimura (1984) in the SISO case, and by Glover (1986) in the MIMO case.

Consider a lightly damped system with varying natural frequency, where even a small variation in natural frequency causes an extremely large variation in the transfer function for each frequency. When the transfer function is described in the fractional form, the numerator (or zero) part varies little, whereas the variation in the denominator (or pole) part is large. The weighting function \( W_1(s) \) given in section 2.2 is convenient to describe the uncertainty of the numerator, but it is not adequate to describe the uncertainty of the denominator. If one is going to describe the uncertainty of the denominator by using only \( W_1(s) \), the stability condition in equation (16) often fails, since the frequency range of \( W_1(s) \) becomes too wide. In this section, a stability condition is developed such that the uncertainty of pole for MIMO system is effectively described.

When \( G(s) \) is a proper real-rational function, consider a factorization

\[ G(s) = D^{-1}(s)N(s), \tag{18} \]

where \( D(s) \) and \( N(s) \) are coprime. For clarity, in this study, when \( D(s) \) and \( N(s) \) are polynomial matrices, this factorization will be called the "matrix fraction description" (Maciejowski, 1989). When the plant shown in Fig. 2 has the matrix fraction uncertainty, the left matrix fraction descriptions of \( G_u(s) \) and \( G'_u(s) \) are obtained as

\[ G_u(s) = [D_u(s) + D_{ud}(s)]^{-1}[N_u(s) + N_{ud}(s)], \tag{19} \]

where \( D_u(s) \) and \( N_u(s) \) are the polynomial matrices with the denominator and the numerator of \( G_u(s) \), respectively, and \( D_{ud}(s) \) and \( N_{ud}(s) \) are the polynomial matrices associated with the uncertainties of \( D_u(s) \) and \( N_u(s) \), respectively. The coprime matrices \( D_u(s) \) and \( N_u(s) \) are real-rational but not proper since they are polynomial matrices whose denominators are 1. Using equation (19), the closed-loop transfer function of equation (11) can be rewritten as

\[ G_{wy}(s) = [I - G_u(s)K(s)]^{-1}G'_w(s) \]

\[ = [I - D_u(s) + D_{ud}(s)][N_u(s) + N_{ud}(s)]^{-1}G'_w(s). \tag{20} \]

Pre-multiplying both equations, the equation in braces {} and the transfer function \( G'_w(s) \), by \( L(s)D_u^{-1}(s)[D_u(s) + D_{ud}(s)] \), yields

\[ G_{wy}(s) = [I - L(s)D_u^{-1}(s)N_{ud}(s)K(s) + L(s)D_u^{-1}(s)D_{ud}(s)]^{-1} \]

\[ L(s)D_u^{-1}(s)[D_u(s) + D_{ud}(s)]G'_w(s). \tag{21} \]

Assuming that the open-loop system is stable, \( D_u^{-1}(s)[D_u(s) + D_{ud}(s)]G'_w(s) \) is also stable. By the small gain theorem, the sufficient stability condition is given as

\[ \left| -L(s)D_u^{-1}(s)N_{ud}(s)K(s) + L(s)D_u^{-1}(s)D_{ud}(s) \right| _{\infty} < 1, \tag{22} \]

which will be called the "matrix fraction stability condition" in this study.

Introduce a positive scalar constant \( \alpha \) and a weighting function \( W_\rho(s) \) which are the upper bounds satisfying

\[ \alpha \left[ N_u(j\omega) \right] > \left[ N_{ud}(j\omega) \right], \]

\[ \left[ W_\rho(j\omega) \right] > \left[ D_u(j\omega)D_{ud}(j\omega) \right] \]

for all \( \omega \). \tag{23}

In this study, \( W_\rho(s) \) will be called the "pole-dependent weighting function". The stability condition in equation (22) can be achieved by

\[ \|aL(s)G_u(s)K(s)\|_{\infty} < 1 - \beta, \]

\[ \|L(s)W_\rho(s)\|_{\infty} < \beta, \quad 0 < \beta < 1. \tag{24} \]

In order that the closed-loop system is robustly stable and satisfies \( \|G_{wy}(s)\|_{\infty} < \gamma \), the mixed-sensitivity problem to satisfy equations (10) and (24) is obtained as

\[ \left\| \frac{L(s)G_u(s)}{1-\beta L(s)G_u(s)K(s)} \right\|_{\infty} < \gamma, \]

\[ \frac{\gamma}{\beta} \left\| L(s)W_\rho(s) \right\|_{\infty} < \gamma, \tag{25} \]

by which a wide stable region can be ensured for a system with varying natural frequency. The inputs and output for the transfer functions in equation (25) are defined as

\[ Z(s) = \left[ \frac{L(s)G_u(s)}{1-\beta L(s)G_u(s)K(s)} \right] \]

\[ \frac{\gamma}{\beta} \left[ W_\rho(s) \right] \]

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\[ \frac{\gamma}{\beta} \left[ W_\rho(s) \right] \]
where \( Z(s) \), \( W(s) \), \( W_1(s) \) and \( W_2(s) \) are the Laplace transformations of \( z(t) \), \( w(t) \), \( w_1(t) \) and \( w_2(t) \), respectively. The equivalent block diagram satisfying equation (26) can be achieved by Fig.3. The relationships between the inputs \( (w, w_1, w_2, u) \) and the outputs \( (z, y) \) are described, as shown in Fig.3, as

\[
\begin{bmatrix}
Z(s) \\
Y(s)
\end{bmatrix} = \begin{bmatrix}
G_w(s) & 0 & \frac{1}{\beta} & \frac{1}{\beta} \\
G_w(s) & \frac{\alpha}{1 - \beta} & 0 & 0
\end{bmatrix} \begin{bmatrix}
W(s) \\
W_1(s) \\
W_2(s) \\
U(s)
\end{bmatrix}
\]

\( U(s) = K(s)Y(s) \),

where \( U(s) \) and \( Y(s) \) are the Laplace transformations of \( u(t) \) and \( y(t) \), respectively.

3. PIEZOELECTRIC-TYPE SERVO-DAMPER

3.1. Structure

Figure 4 shows the schematic of the computer-controlled servo-damper system. The beam, which simulates a flexible manipulator, is driven back and forth by a stepping motor such that the beam length \( L \) can range from 0.5 to 0.67 meters. Two accelerometers, mounted perpendicularly to each other near the free end of the beam, measure the horizontal and vertical accelerations which are double-integrated through the charge amplifiers to give the corresponding displacements. Band-pass analog filters are also incorporated to eliminate the low-frequency floating error due to double integration and high-frequency noise.

In order to suppress the vibration at the free end of the beam, a piezoelectric-type servo-damper is employed which serves as an active damper. Note that the control device is made small and compact, and that the control force is imposed directly onto the free end. Figure 5 shows the front view of the piezoelectric-type servo-damper, which consists of a fixture and a sprung mass supported by two pairs of stack-type piezoelectric actuators placed perpendicularly to each other and pre-strained by bolts. While this actuator has some drawbacks in practical usage such as small displacement \( (15 \mu m/100V) \), hysteresis and low tensile strength, it has advantages such as low driving voltage (below 100V), fairly large force generation (max. 350kgf/100V), rapid response time, wide dynamic range and compactness.

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Though the cantilever beam with a servo-damper is designed to be symmetric, small coupling effects happen between vibrations in the x and y directions. \( \lambda_1 \ll 1 \) and \( \lambda_2 \ll 1 \) are the upper bounds of coupling ratios in the x and y directions, respectively. Using the approximate relationships of \( \Delta L_x(t) \propto u_x(t) \) and \( \Delta L_y(t) \propto u_y(t) \) where \( u_x(t) \) and \( u_y(t) \) are the supplied voltages, and assuming that \( M_{eq}(t), C_{eqx}(t), C_{eqy}(t), K_{eqx}(t) \) and \( K_{eqy}(t) \) are slowly time-varying, equation (28) can be rewritten, in the Laplace domain, as

\[
M_1(s) = \begin{bmatrix}
X_b(s) \\
Y_b(s) \\
X_d(s) \\
Y_d(s) \\
W_x(s) \\
W_y(s)
\end{bmatrix} = M_2(s) \begin{bmatrix}
W_x(s) \\
W_y(s) \\
U_x(s) \\
U_y(s)
\end{bmatrix},
\]

(29)

where \( K_L(15\text{um}/100\text{V}) = \Delta L_x(t)/u_x(t) = \Delta L_y(t)/u_y(t) \) is the proportionality constant, and where \( X_b(s), Y_b(s), X_d(s), Y_d(s), W_x(s), W_y(s), U_x(s) \) and \( U_y(s) \) are the Laplace transformations of \( x_b(t), y_b(t), x_d(t), y_d(t), W_x(t), W_y(t), U_x(t) \) and \( U_y(t) \), respectively. \( M_1(s) \) and \( M_2(s) \) are obtained as:

\[
M_1(s) = \begin{bmatrix}
A_{Mx} + A_x & \lambda_1 A_y + \lambda_2 A_x - A & -\lambda_2 A \\
\lambda_1 A_y + \lambda_2 A_x + A_{Mx} + A & A_{Mx} + A & -\lambda_2 A \\
-\lambda_2 A & -\lambda_2 A & -\lambda_2 A + A \\
\lambda_2 A & -\lambda_2 A & -\lambda_2 A + A
\end{bmatrix}
\]

(30)

where

\[
\begin{align*}
A_{Mx} &= M_{eq}^2 + C_{eqx}s + K_{eqx}, \\
A_{My} &= M_{eq}^2 + C_{eqy}s + K_{eqy}, \\
A_x &= C_{eqx}s + K_{eqx}, \\
A_y &= C_{eqy}s + K_{eqy}, \\
A &= cs + k.
\end{align*}
\]

Pre-multiplying equation (29) by \( M_1^{-1}(s) \),

\[
\begin{bmatrix}
X_b(s) \\
Y_b(s) \\
X_d(s) \\
Y_d(s) \\
W_x(s) \\
W_y(s)
\end{bmatrix} = \frac{\text{adj} M_1(s)}{\det[M_1(s)]} \begin{bmatrix}
W_x(s) \\
W_y(s) \\
K_L U_x(s) \\
K_L U_y(s)
\end{bmatrix},
\]

(31)

Assuming \( \lambda, \lambda_j \approx 0, i,j = 1,2 \), determinant of \( M_1(s) \) is

\[
\det[M_1(s)] \approx \lambda^2(A_{Mx} + ms^2)(A_{My} + ms^2)
\]

(32)

which can be reduced to

\[
\det[M_1(s)] \approx \lambda^2(A_{Mx} + ms^2)(A_{My} + ms^2)
\]

(33)
over the low-frequency range (typically below 10–20 Hz) since the term \( A \) (\( = cs + k \)) is considerably greater than other parameters. Using this approximation and accounting for the first-order band-pass filter dynamics, the equations of motion for \( X_b(s) \) and \( Y_b(s) \) are given by

\[
Y(s) = G_w(s)W(s) + G_u(s)U(s), \quad (34)
\]

where

\[
Y(s) = \begin{bmatrix} X_b(s) \\ Y_b(s) \end{bmatrix}, \quad W(s) = \begin{bmatrix} W_x(s) \\ W_y(s) \end{bmatrix}, \quad U(s) = \begin{bmatrix} U_x(s) \\ U_y(s) \end{bmatrix},
\]

\[
G_w(s) = \begin{bmatrix} (C_{eqx} + K_{eqx})T_x(s) \\ (C_{eqy} + K_{eqy})T_y(s) \end{bmatrix}, \quad \lambda_1(C_{eqx} + K_{eqx})T_x(s) + \lambda_2(C_{eqy} + K_{eqy})T_y(s),
\]

\[
G_u(s) = \begin{bmatrix} -ms^2K_xT_x(s) \\ -ms^2K_yT_y(s) \end{bmatrix} - \begin{bmatrix} \lambda_1ms^2K_xT_x(s) \\ \lambda_2ms^2K_yT_y(s) \end{bmatrix},
\]

\[
T_x(s) = \frac{\omega_{x1}^2}{(s + \omega_{x1})(s + \omega_{x2})(M_{eq} + ms^2 + C_{eqx} + K_{eqx})},
\]

\[
T_y(s) = \frac{\omega_{y1}^2}{(s + \omega_{y1})(s + \omega_{y2})(M_{eq} + ms^2 + C_{eqy} + K_{eqy})},
\]

where \( \omega_{x1} \), \( \omega_{x2} \), \( \omega_{y1} \), and \( \omega_{y2} \) are the high- and low-pass filter frequencies in the \( x \) and \( y \) directions, respectively.

4. EXPERIMENTS

In this section, the proposed \( H_\infty \) controller, based upon the matrix fraction stability condition and the pole-dependent weighting function, is applied to a moving beam with a piezoelectric-type servo-damper in order to reduce the operational vibration at its free end. The system identification procedure is firstly described. After the controller design procedure is outlined, the performance and robust stability of the \( H_\infty \) controller are investigated through the control experiments.

In order to identify the system parameters, modal testings are performed separately for the horizontal and vertical directions when the beam length takes the minimal (0.5 m), nominal (0.585 m), and maximal (0.67 m) values. The random input generated from a random signal generator drives a pair of piezoelectric actuators and the displacement is measured by using an accelerometer. The measured frequency response function (FRF) between the random input and the beam displacement is zoom transformed near the first natural frequency using a signal analyzer, and then curve fitted. Figure 7 shows the FRFs of the open-loop system \( G_w(s) \) for each length of the beam. It is found that the first natural frequencies vary from 9.7 to 10.4 Hz in \( x \) direction, and from 15.3 to 19.0 Hz in \( y \) direction. The fitted nominal transfer functions \( G_w(s) \) and \( G_u(s) \) are obtained from the modal testing for the nominal length of the beam, whereas the constant \( \alpha \) and the pole-dependent weighting function \( W_p(s) \) are obtained from the variations in the numerator and denominator of the transfer functions \( G_u(s) \) when the beam length varies from the minimal to the maximal. The upper bounds of the coupling ratios, \( \lambda_1 \) and \( \lambda_2 \), are found to be 0.05.

The coefficients of the transfer functions are determined as follows:

\[
G_w(s) = \begin{bmatrix} (0.0920s + 4005)T_x(s) \\ (0.023s + 568)T_x(s) \end{bmatrix}, \quad \begin{bmatrix} (0.0046s + 2000)T_y(s) \\ (0.462s + 11360)T_y(s) \end{bmatrix},
\]

\[
G_u(s) = \begin{bmatrix} -0.0326s^2T_x(s) - 0.020s^2T_x(s) \\ -0.016s^2T_y(s) - 0.0407s^2T_y(s) \end{bmatrix},
\]

where

\[
T_x(s) = \frac{628s}{(s + 33.8)(s + 628)(s^2 + 0.920s + 4005)},
\]

\[
T_y(s) = \frac{628s}{(s + 48.0)(s + 628)(s^2 + 0.462s + 11360)}.
\]

Fig. 7. FRFs of the open-loop system
\[ N_p(s) = 628s^2 - 0.020s^2 \]
from which \[ \overline{W}_p(s) \] is given as
\[ \overline{W}_p(s) = \begin{bmatrix} 0.0213s + 299 & 0 \\ 0 & 0.0691s + 2930 + 4005 \end{bmatrix} \]

The results from modal testing show that filter frequencies, \( \omega_{x1}, \omega_{y1}, \omega_{x2} \) and \( \omega_{y2} \) are tuned to 5.4, 7.6, 100 and 100 Hz, respectively. The transfer functions in equations (35) and (37) have the realization forms
\[ G_w(s) = C_w(sI - A_w)^{-1} B_w, \]
\[ G_u(s) = C_u(sI - A_u)^{-1} B_u, \]
\[ \overline{W}_p(s) = C_p(sI - A_p)^{-1} B_p, \]

where \( A_w, A_u, B_w, B_u, B_p, C_w \) and \( C_p \) are the system matrices of \( G_w(s), G_u(s) \) and \( \overline{W}_p(s) \). By substituting equation (38) into equation (27), each matrix in equation (1) can be determined as:
\[ A = [A_w \ 0 \ 0; 0 \ A_u], \]
\[ B_1 = [B_w \ 0 \ 0; 0 \ 0 \ B_p], \]
\[ B_2 = [B_u \ 0], \]
\[ C_1 = [C_w \ \chi \ C_p], \]
\[ C_2 = [C_u \ \chi \ C_p], \]
\[ D_{11} = [0 \ 0 \ 0; 0 \ 0 \ 0], \]
\[ D_{12} = [0], \]
\[ D_{21} = [0 \ \gamma \ 0; 0 \ 1 - \beta \ 0]. \]

The desired maximal magnitude \( \gamma \) of the FRFs of the closed-loop system \( G_{wy}(s) \) is given as 20, which is estimated to be 3.3% and 9.6% of the maximal magnitudes of \( G_x(s) \) in the x and y directions, respectively. The values of \( \alpha \) and \( \beta \) are given as 0.03 and 0.9, respectively. From Steps 1 ~ 4 given in section 2.1, \( J \) is determined as
\[ J = \begin{bmatrix} 3.2 \times 10^7 & 0 \\ 0 & 2.5 \times 10^8 \end{bmatrix} \]

The system matrices of the \( H_\infty \) controller are determined using equation (7). Figure 8 shows that all FRFs from external inputs \( (w, w_1, w_2) \) and the control output \( z \) remain less than \( \gamma \).

To investigate the performance and robust stability of the designed \( H_\infty \) controller, the control experiments are performed for the cantilever beam with a piezoelectric-type servo-damper. The sampling time is given as 2.2 msec and the control input voltage driving the actuators is limited between 0 and 100 volts. Figures 9 and 10 show the experimental time histories with and without control at the free end of beam when initial displacements are given for each length of the beam. Figures 11 and 12 show the experimental time histories without and with control at the free end of the beam when the beam is driven from maximal length to minimal at speed of 17 mm/sec for 10 seconds after starting at maximal length. These figures show that the system with the proposed \( H_\infty \) controller has a good performance and it is stable for all lengths of the beam.

5. CONCLUSIONS

The matrix fraction stability condition and the pole-dependent weighting function are defined for plants with uncertainty. For \( H_\infty \) output feedback control, the mixed sensitivity problem satisfying the matrix fraction stability condition is formulated. The \( H_\infty \) controller is designed to have a good robust stability and performance for a system whose poles have large variations, i.e., whose natural frequency varies widely. The proposed \( H_\infty \) controller is then applied to a cantilever beam with a piezoelectric-type servo-damper which serves as an active damper. The experimental results for the operational vibration control reveal that the proposed \( H_\infty \) controller design scheme and the piezoelectric-type servo-damper can be effectively used to suppress the vibration of a system with varying natural frequency.
6. REFERENCES


